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THE FORMATION OF ZERO FREQUENCY INTERNAL WAVES DURING FREE CONVECTION IN A TEMPERATURE-STRATIFIED LIQUID*

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It is shown, as a result of an analysis of the equations of free convection in a temperature-stratified medium (TSM), that internal waves of zero frequency are formed when a thermal source is included together with a floating flare. The wavelength of these waves is calculated and the parameters of the transition processes are determined.

Zero-frequency internal waves, which are observed experimentally /1/, are an important element of convective flows which are generated by thermal sources in liquids with saline stratification. Only the parameters of a flare which floats above a localized source of heat have been calculated in a TSM /2/. There is interest in the possibility of the existence of zero-frequency internal waves which are excited by the thermal source in a TSM and in determining their parameters.

1. Formulation of the problem. The linearized system of convection equations in the TSM in a cylindrical system of coordinates at the origin of which a thermal source with a power P is located and where the gravitational vector $\mathbf{g}$ is directed opposite to the z-axis has the form

$$\partial\mathbf{u}/\partial t = -\frac{1}{\rho_0}\nabla p + \nu\Delta\mathbf{u} + \frac{\nu}{3}\nabla(\nabla\cdot\mathbf{u}) - \alpha T'\mathbf{g} \tag{1.1}$$

$$\partial T'/\partial t + \nabla\cdot(\mathbf{u}T_0(z)) = \chi\Delta T' + \frac{P}{c_p\rho_0}\frac{\delta(z)\,\delta(r)}{2\pi r}\theta(t)$$

$$\partial\rho/\partial t - \alpha\rho_0\nabla\cdot(\mathbf{u}T_0(z)) + \rho_0\nabla\cdot\mathbf{u} = -\frac{\alpha P}{c_p}\frac{\delta(z)\,\delta(r)}{2\pi r}\theta(t)$$

$$\rho = \rho_0(1-\alpha T),\quad T = T_0(z) + T',\quad T_0(z) = T_0(1 + z/(\alpha T_0\Lambda))$$

Here $\mathbf{u}$ is the velocity of the medium, p is the pressure after subtracting the hydrostatic pressure, $T, T_0(z)$ and T' are the total, stratifying and excess temperatures, T_0 and ρ_0 are the temperature and density of the medium at the level $z = 0$, ρ is the density of the medium, α, χ and ν are the coefficients of thermal expansion, the thermal diffusivity and the kinematic viscosity, c_p is the heat capacity of the medium at constant pressure and Λ is the temperature stratification scale. The initial and boundary conditions, taken at infinity and the conditions on the functions $\mathbf{u}, p$ and T' are homogeneous.

The velocity field, which is axially symmetric can be represented in the form

$$u = \mathbf{v} + \mathbf{w},\quad \mathbf{v} = -\nabla h,\quad w_r = -\partial\Phi/\partial r,\quad w_z = -\partial\Psi/\partial z$$

$$\Delta_r\Phi + \frac{\partial^2\psi}{\partial z^2} = 0,\quad \Delta_r = r^{-1}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)$$

Here, w_r and w_z are the radial and vertical components of the solenoidal part of the

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velocity and h, Φ and Ψ are unknown functions of the coordinates and time.

The components of the total velocity vector can be written in the form

$$u_r = -\frac{\partial h}{\partial r} - \frac{\partial^2 f}{\partial r \partial z}, \quad u_z = -\frac{\partial h}{\partial z} + \Delta_r f, \quad f = \int_0^z \Phi(r, \zeta, t)\, d\zeta \tag{1.2}$$

Substitution of (1.2) and (1.1) enables one to reduce the initial system to the system

$$\left(\frac{\partial}{\partial t} - \chi\Delta\right)\left(\frac{\partial}{\partial t} - \nu\Delta\right)\Delta f + N^2 \Delta_r f = Q\,\frac{\delta(z)\,\delta(r)}{2\pi r}\,\theta(t) \tag{1.3}$$
$$h = -\frac{\chi}{g}\,\Delta\left(\frac{\partial}{\partial t} - \nu\Delta\right) f, \quad N^2 = \frac{g}{\Lambda}, \quad Q = \frac{g\alpha P}{c_p \rho_0}$$

By applying a Fourier-Bessel transformation of the form

$$F(k_r, k_z, \omega) = \frac{1}{2\pi}\int_0^\infty \left\{\int\!\!\int_{-\infty}^{+\infty} f(r, z, t) \exp(ik_z z - i\omega t)\, dt\, dz\right\} r J_0(k_r r)\, dr$$

to the first equation of (1.3), where J_0 is a zero order Bessel function of the first kind, we find the image of the function $f(r, z, t)$ and, by applying the inverse transformation, we get

$$f(r, z, t) = \frac{Q}{(2\pi)^3}\int_0^\infty \left\{\int\!\!\int_{-\infty}^{+\infty} \frac{(\pi\delta(\omega) - i\,\mathrm{Vp}(1/\omega)) \exp(i\omega t - ik_z z)\, d\omega\, dk_z}{k^2(\omega^2 - i(\nu + \chi)k^2\omega - \nu\chi k^4 - N^2 k_r^2/k^2)}\right\} \times \tag{1.4}$$
$$k_r J_0(k_r r)\, dk_r; \quad k^2 = k_r^2 + k_z^2$$

The properties of the function $\mathrm{Vp}(1/\omega)$ are defined by the relationship

$$\int_{-\infty}^{+\infty} \varphi(\omega)\, \mathrm{Vp}\left(\frac{1}{\omega}\right) d\omega = \lim_{\varepsilon \to +0}\left(\int_{-\infty}^{-\varepsilon} \frac{\varphi(\omega)}{\omega}\, d\omega + \int_{\varepsilon}^{+\infty} \frac{\varphi(\omega)}{\omega}\, d\omega\right)$$

After integration with respect to the variable ω, (1.4) takes the form

$$f(r, z, t) = (2\pi)^{-2} Q\,(I_0 + I_+ + I_-) \tag{1.5}$$
$$I_0 = -\theta(t)\int_0^\infty \left\{\int_{-\infty}^{+\infty} \frac{\exp(-ik_z z)\, dk_z}{k^2(\nu\chi k^4 + N^2 k_r^2/k^2)}\right\} k_r J_0(k_r r)\, dk_r$$
$$I_\pm = \pm\frac{1}{2}\int_0^\infty \left\{\int_{-\infty}^{+\infty} F_\pm(k_r, k_z, t) \exp(-ik_z z)\, dk_z\right\} k_r J_0(k_r r)\, dk_r$$
$$F_\pm(k_r, k_z, t) = (i(\nu + \chi)k^2/2 \pm M)^{-1} M^{-1} \exp(-(\nu + \chi)k^2 t/2 \pm iMt)$$
$$M = (N^2 k_r^2/k^2 - (\nu - \chi)^2 k^4/4)^{1/2}$$

2. ***Asymptotic estimate of*** $I_\pm$. To estimate the behaviour of $I_\pm$ at large values of the time t the integrand is first reduced to a form which is convenient for the application of asymptotic methods. In order to do this, a coordinate transformation of the form

$$k_r^2 = \eta^2 p^2 q^2 V^2(p, q), \quad k^2 = V^2(p, q) = p^2/\chi + q^2/\nu,$$
$$\eta = (\nu - \chi)/(N(\nu\chi)^{1/2})$$

is carried out in the region of $\operatorname{Im} M = 0$ while the transformation

$$k_r^2 = y^2 H(x, y), \quad k^2 = y^2(1 - H(x, y)), \quad H(x, y) = (x^2 + (\nu - \chi)^2 y^4/4)/N^2$$

is used in the region $\operatorname{Im} M \neq 0$, where p, q, x and y are new variables. As a result of this, we get

$$I_\pm = \pm\frac{2\eta^2}{\nu - \chi}\left\{\int_0^{a\sqrt{\nu}}\int_0^{b(q)} + \int_{a\sqrt{\nu}}^{+\infty}\int_0^{c(q)}\right\} \frac{\exp\{-(p^2\xi_\pm + q^2/\xi_\pm)t\}}{p^2\xi_\pm + q^2/\xi_\pm}\, A(p, q)\, dp\, dq\, \pm \tag{2.1}$$
$$\frac{1}{N^2}\int_0^{a\sqrt{2}}\int_0^{X(y)} \exp(-(\nu + \chi)y^2 t/2 \pm ixt)\, B_\pm(x, y)\, dx\, dy$$

$$A(p,q)=\frac{\cos(zV(p,q)(1-\eta^2p^2q^2)^{1/2})J_0(r\eta pqV(p,q))pq}{V(p,q)(1-\eta^2p^2q^2)^{1/2}}$$

$$B_\pm(x,y)=\frac{\cos(zy(1-H(x,y))^{1/2}J_0(ryH^{1/2}(x,y))}{(i(\nu+\chi)y^2/2\pm x)(1-H(x,y))^{1/2}}$$

$$\xi_+=\frac{\nu}{\chi},\quad \xi_-=1,\quad a=(N/(\nu-\chi))^{1/2},\quad b(q)=q\xi_+^{-1/2}$$

$$c(q)=(\eta q)^{-1},\ X(y)=(N^2-(\nu-\chi)^2y^4/4)^{1/2}$$

Two different oscillating functions, a cosine and a Bessel function, occur in the integrand of (2.1). On account of this, relationships (2.1) determine the different asymptotic behaviour of $I_\pm$ in the different regions of space and time.

In the case of the integrals of $A(p,q)$ it is advisable to introduce "small" and "large" distances which are defined by the relationships

r, z are "small" if $r, z \ll \chi at$
r, z are "large" if $r, z \gg \nu at$

The following relationships hold in the case of $B_\pm$:

r, z are "small" if $r, z \ll (\nu+\chi)at/\sqrt{2}$;
r, z are "large" if $r, z \gg (\nu+\chi)at/\sqrt{2}$.

By using the method of steepest descent and the stationary phase method /4/ and the representation (1.2), it is possible to obtain estimates for the components of the velocity **u** in the different spatial regions.

The region of "small" distances $r, z \ll (\nu+\chi)a\sqrt{2}t$

$$u_r \sim Q(-Drz + W(\nu,\chi,N)\varepsilon Rt^{-2}\exp(-\varkappa R)J_1(\varkappa R)) \tag{2.2}$$

$$u_z \sim {}^1/_2 Q(Dz^2 + W(\nu,\chi,N)\varkappa\varepsilon^2R^2t^{-2}(1+\varepsilon^2)^{-1/4}\exp(-\varkappa R)(J_0(\varkappa,R) - J_1(\varkappa R)/(\varkappa R)))$$

$$D=(\nu-\chi)^4N^{-4}t^{-3/2}\nu^{-7/2}\chi^{-3}/4$$

$$\varkappa=\left(\frac{N(1+\varepsilon^2)}{4\nu\varkappa}\right)^{1/4},\quad R^2=r^2+z^2,\quad \varepsilon=\frac{z}{Ntr}$$

Here, $W(\nu,\chi,N)$ is a function of ν,χ and N.

The first term in (2.2) is only important in the case of subcritical flow regimes in the region $r, z \ll \chi t({}^1/_2 N/(\nu\chi)^{1/2})^{1/2}$. The second term together with the asymptotic formulae for I_0 describe the transition process of the formation of zero-frequency waves. Both terms in (2.2) decay rapdily with time.

The region of "small" radial and "large" vertical distances $r \ll \chi ta,\ z \gg (\nu+\chi)t\ a/\sqrt{2}$

$$u_r \sim rz^{-1}Q(t)\cos(az) \tag{2.3}$$

$$u_z \sim -2Q(t)\,\mathrm{ci}(az)$$

$$Q(t)=\frac{4\pi Q}{(N\nu\chi(\nu-\chi))^{1/2}}\left(\frac{\nu}{\nu+\chi}\exp(-\nu a^2t)-\exp(-\chi a^2t)\right)$$

The structure of the flow which is described by relationships (2.3) consists of segments of stationary cells inclined at an angle to the z-axis which are repeated along the vertical with a period $2\pi a$. The intensity of the velocity field in the cells decreases with height and decays expontially with time. At the same time, the velocities on the boundaries of adjacent cells are in opposite directions.

Numerical calculations of the vertical size H of a cell gave the following results.

Air: $\nu = 1.3\ \ 10^{-5}\ \mathrm{m^2/s}$, $\chi = 2\times10^{-8}\ \mathrm{m^2/s}$, $N = 0.01$ Hz, $H = 2.3\times10^{-1}$ m.

Water: $\nu = 10^{-6}\ \mathrm{m^2/s}$, $\chi = 1.5\times10^{-7}\ \mathrm{m^2/s}$, $N = 0.01$ Hz, $H = 6\times10^{-2}$ m.

The region of "large" radial and "small" vertical distances $r \gg (\nu+\chi)ta,\ z \ll \chi ta$

$$u_r \sim \frac{zQ}{r^2N}\left(\frac{\nu-\chi}{(\nu+\chi)^2}+\left(\frac{\pi r}{2}\right)^{1/2}(2a^2)^{3/4}\exp(-(\nu+\chi)a^2t)J_0(ar)\right) \tag{2.4}$$

$$u_z \sim \frac{z^2Q}{2r^3N}\left(\frac{\nu-\chi}{(\nu+\chi)^2}+\left(\frac{\pi r}{2}\right)^{1/2}(2a^2)^{3/4}\exp(-(\nu+\chi)a^2t)\times\right.$$
$$\left.\left(\frac{J_0(\sqrt{2}ar(2a)^{1/2})}{2r}+\sqrt{2}aJ_1(r(\sqrt{2}a)^{1/2})\right)\right)$$

The flow which is described by relationships (2.4) is of low intensity, has a high order of decrease in the radial direction, and decays rapidly with time. This is due to the fact that, in this region of space, we did not succeed in obtaining either substantial overheating or a zero-frequency wave.

3. *Structure of the zero frequency waves.* The structure of the flow of the zero frequency waves is described by the integral I_0. Its asymptotic behaviour is investigated at large distances from the heat source along a selected direction which is characterized by the magnitude of γ, the tangent of the angle to the horizontal, and R, the distance from the source.

The method of steepest descent /4/ and the representation (1.2) yield estimates of the radial and vertical components of the velocity u when

$$R \gg (2\nu\chi/N^2)^{1/4} \tag{3.1}$$

$$u_r \sim U(\gamma, R)\sin(\Lambda - \pi/4), \quad u_z \sim -\gamma U(\gamma, R)\cos(\Lambda - \pi/4)$$

$$U(\gamma, R) = \frac{\gamma Q\theta(t)}{8\pi N R(\nu\chi)^{1/2}}\exp(-\Lambda); \quad \Lambda = \left(\frac{N^2(1+\gamma^2)}{4\nu\chi}\right)^{1/4} R$$

Zero-frequency oscillations are also generated by the function h which describes the potential part of the velocity. The complex expression which was calculated is not presented here since the ratio of the potential and solenoidal parts of the velocity in the aymptotic region is proportional to $\nu\chi\,(N^2/(4\nu\chi))^{3/4}/g \ll 1$ and the contribution from the function h can be neglected.

Relations (3.1) describe zero-frequency waves (stationary waves /5/), the nature of which is determined by the combined action of kinetic phenomena and the effect of buoyancy. The dependence of the emission wavelength on ν, χ and N is a consequence of this:

$$\lambda = 2\pi\,(4\nu\chi\,(1+\gamma^2)/N^2)^{1/4}$$

The form of relationships (3.1) indicates that the energy density flux vector $\mathbf{q} = \rho\mathbf{u}\,(u^2/2 + c_vT' + p/\rho)$ attains its greatest absolute value when γ is close to zero, which determines an almost horizontal propagation of the zero-frequency internal waves at large distances from the source.

It follows from (3.1) that the trajectories of the particles participating in the zero-frequency oscillations are ellipses with an eccentricity which increases on passing farther away from the source while the inclination of the semimajor axis to the horizontal tends to zero.

Relationships (3.1) determine the high sensitivity of the characteristics of these waves to the parameters of the medium. For instance, the waves decay more rapidly and their wavelength increases in the region with more pronounced temperature gradients. It also follows from (3.1) that, on passing through the boundary of separation of two regions with different temperature gradients, the zero-frequency waves are partially reflected and break up, which is indicative of their instability with respect to local superheating.

In practical physical situations, when the powers of the thermal source are greater than the critical power, convective motion of the medium occurs close to the heat source, as a result of which the structure of the flow cannot be described by Eqs.(1.3). Meanwhile, this convection region is bounded, and beyond its confines the solutions of the linearized problem are applicable, as has also been considered in /6/.

Under subcritical flow conditions, that is, when the power of the source is less than the critical power /1/, the solutions of (1.3) are applicable over the whole of the space.

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